3.9 measles model with vaccines

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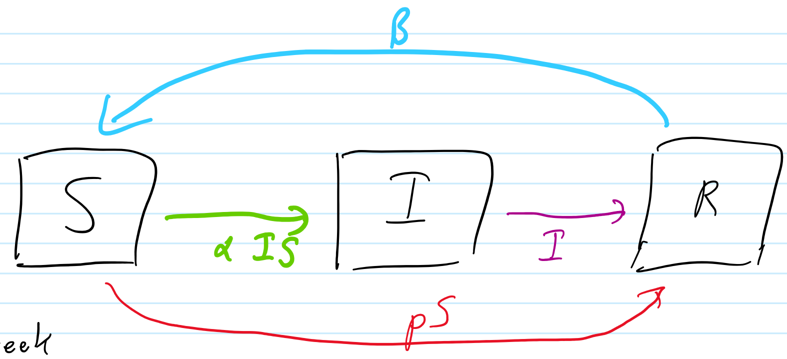
parameters include proportions



$N = S + I + R$

Time-intervals = 1 week

Assumptions: (1) Each week, infected recover and are immune

(2) Constant number of deaths of recovered patients β

(3) New infected individuals $= \alpha I S$

(4) Immunization of p fraction of population (pS)

$$S_{t+1} = (1-p)S_t - \alpha I_t S_t + \beta$$

$$I_{t+1} = \alpha I_t S_t$$

$$R_{t+1} = R_t + I_t - \beta + pS_t$$

simplification of deaths

Note: $N_t \equiv N$ a constant, and $N = S_t + I_t + R_t$, so $R_t = N - (I_t + S_t)$

$$\begin{cases} S_{t+1} = (1-p)S_t - \alpha I_t S_t + \beta \\ I_{t+1} = \alpha I_t S_t \end{cases}$$

Solve: $\begin{cases} \bar{S} = (1-p)\bar{S} - \alpha \bar{I}\bar{S} + \beta \\ \bar{I} = \alpha \bar{I}\bar{S} \end{cases}$ to find equilibria

disease-free equilibrium

Case 1: $\bar{I} = 0$, so $p\bar{S} = \beta \Rightarrow \bar{S} = \frac{\beta}{p}$, $p > 0$.

Case 2: $\bar{I} \neq 0$, so $1 = \alpha \bar{S} \Rightarrow \bar{S} = \frac{1}{\alpha}$

$$\Rightarrow \frac{1}{\alpha} = (1-p)\cdot\frac{1}{\alpha} - \bar{I} + \beta$$

$$\Rightarrow \frac{p}{\alpha} = -\bar{I} + \beta$$

$$\Rightarrow \bar{I} = \beta - \frac{p}{\alpha}, \quad \bar{S} = \frac{1}{\alpha},$$

endemic equilibrium.

Need $\beta > \frac{p}{\alpha}$ (positivity)

Compute Jacobian:

$$\begin{cases} S_{t+1} = (1-p)S_t - \alpha I_t S_t + \beta & = f(S_t, I_t) \\ I_{t+1} = \alpha I_t S_t & = g(S_t, I_t) \end{cases}$$

where $f(S, I) = (1-p)S - \alpha I S + \beta$

$g(S, I) = \alpha I S$

$$J(S, I) = \begin{bmatrix} \frac{\partial f}{\partial S} & \frac{\partial f}{\partial I} \\ \frac{\partial g}{\partial S} & \frac{\partial g}{\partial I} \end{bmatrix} = \begin{bmatrix} 1 - p - \alpha I & -\alpha S \\ \alpha I & \alpha S \end{bmatrix}$$

Disease-free equilibrium:

$$J\left(\frac{\beta}{p}, 0\right) = \begin{bmatrix} 1-p & -\frac{\alpha\beta}{p} \\ 0 & \frac{\alpha\beta}{p} \end{bmatrix}$$

$$\lambda_1, \lambda_2 = \underbrace{1-p}_{<1 \text{ if } p \text{ is positive}}, \quad \frac{\alpha\beta}{p}.$$

So if $\alpha\beta < p$, then eigenvalues have modulus < 1, so the disease-free equilibrium is locally asymptotically stable.

Let $R_0 = \frac{\alpha\beta}{p}$, the basic reproduction number.

If $R_0 < 1$, then the disease-free equilibrium is locally asymp. stable.

If $R_0 > 1$, then the disease-free equilibrium is unstable.

Endemic equilibrium: $J\left(\frac{1}{\alpha}, B - \frac{p}{\alpha}\right) = \begin{bmatrix} 1 - \alpha B & -1 \\ \alpha B - p & 1 \end{bmatrix}$

By the Jury conditions, this is locally asymptotically stable

if $$|2 - \alpha B| < \underbrace{2 - p < 2}_{\text{always true if } p > 0.}$$

If $\alpha B < 2$, then stability criteria hold if $1 < R_0 < \frac{2}{p}$.

Let's simulate some trajectories.

Let $B = 115$, $\alpha = 3 \cdot 10^{-5}$, $p = 0$, 0.002, or 0.005,

$N = 5 \cdot 10^6$, $S_0 = 3 \cdot 10^4$, $I_0 = 200$ [Fig. 3.15, book] [Anderson & May, 1982]

$$J\left(\frac{1}{\alpha}, B - \frac{p}{\alpha}\right) = \begin{bmatrix} 1 - \alpha B & -1 \\ \alpha B - p & 1 \end{bmatrix}$$

$$\lambda_1, \lambda_2 = \frac{2 - \alpha B \pm \sqrt{\alpha^2 B^2 - 4\alpha B + 4p}}{2}$$

When the square root is imaginary, $\lambda_1 = \bar{\lambda}_2$ are complex conjugates.

$$|\lambda_i|^2 = \lambda_1 \lambda_2 = \frac{1}{4}\left[4 - 4\alpha B + \alpha^2 B^2 - (\alpha^2 B^2 - 4\alpha B + 4p)\right]$$
$$= \frac{1}{4}[4 - 4p] = 1 - p.$$

So for $\rho = 0$, and $\alpha\beta < 4$, λ_1, λ_2 have

modulus 1 and are complex conjugates,

explains why the endemic eq was not asymp. stable.